then Equation (10) gives

$$a^{(3)}(t) = e^{Kt} a^{(3)}(0) = e^{Kt} \{r\, a^{(1)}(0) + (1-r)a^{(2)}(0)\} = ra^{(1)}(t) + (1-r)a^{(2)}(t) \quad (20)$$

which is of the linear form of Equation (19).

SUMMARY AND CONCLUSION

It is possible to uncouple any first-order reaction system into independent reactions. These independent reactions represent straight-line reaction paths and are directly measurable. An expression, in terms of compositions along the straight-line paths, is given for all reaction paths and is useful in the simulation of chemical processes.

NOTATION

A_i = the i^{th} chemical species
B_j = the j^{th} characteristic species
a_i = concentration of species A_i, mole/volume
b_j = concentration of species B_j, mole/volume
c_{ij} = constants in Equation (2), mole/volume
h_i = height of fluid in column A_i, length
k_{ij} = kinetic rate constant from species A_j to A_i, 1/time
t = time
λ_i = characteristic decay constant, 1/time

Vectors

a = composition expressing concentration of A species
b = composition expressing concentration of B species
X_i = characteristic vector of matrix K

Matrices

D = diagonal matrix of equilibrium composition
K = rate constants
S = a symmetric matrix
X = matrix of characteristic vectors of K
Λ = diagonal matrix of characteristic values

Superscripts and subscripts

$*$ = equilibrium value
o = initial value
T = transpose of a vector or matrix

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Pressure Drops in the Flow of Gases Through Packed and Distended Beds of Spherical Particles

CHARLES A. WENTZ, JR., and GEORGE THODOS

The Technological Institute, Northwestern University, Evanston, Illinois

Pressure drop measurements were made across packed and distended beds having five layers of smooth plastic spheres, 1.23 in. in diameter, arranged in cubic, body-centered cubic, and face-centered cubic orientations of void fractions varying from 0.354 to 0.882. The distended beds were prepared by separating the spheres with permanently attached short lengths of fine rigid wire. Friction factors were calculated from the overall pressure drop measurements with the Ergun equation and were plotted against the corresponding modified Reynolds number which ranged from 2,550 to 64,900. A single relationship resulted for both types of beds which is independent of the geometric orientation and void fraction of the spheres of the bed.

To eliminate entrance and exit effects of the air flowing through the bed pressure drop measurements were also made across the middle layer of each distended bed. Again a single relationship between the corresponding friction factors and modified Reynolds number was obtained which is independent of the geometric orientation and void fraction of the bed.

From the two relationships between friction factors and modified Reynolds number the ratio f'_k/f_k for a packed or distended bed having five layers of spheres is 1.13. This ratio should decrease with increasing number of layers of spheres and approach the limiting value of 1 when the number of layers becomes very large.

Packed columns of granular solids are widely used in many mass and heat transfer operations such as extraction and distillation. Measurements of pressure drops for the flow of gases through packed beds have been made by many investigators (*1, 2, 3, 4, 5, 6, 8*), and relationships between the pressure drop and the Reynolds number have been developed.

Recently McConnachie (*7*) has investigated heat, mass, and momentum transfer for the flow of air past spheres arranged in a body-centered cubic orientation and held apart in space by short lengths of fine rigid wire to produce distended fixed beds having void fractions comparable to those of fluidized beds. Although he conducted pressure drop measurements, experimental limitations did not permit accurate comparisons between the resulting values and corresponding measurements for packed beds. Therefore an attempt has been made in the present study to investigate the effect of the geometric orientation of the particles of the bed on the pressure drop and to determine if the existing relationships between the pressure drop and Reynolds number for packed beds can be extended to distended beds.

EXPERIMENTAL EQUIPMENT AND PROCEDURE

The experimental apparatus included a vertical wind tunnel, 14 in. in diameter

Table 1. Experimental Data and Derived Quantities for Typical Pressure Drop Runs

Run no.	u, ft./sec.	$\rho u^2/2g_c$	$D_pG/\mu(1-\epsilon)$	Total bed (5 layers) ΔP, lb.f/sq. ft.	f'_1	f'_k	Test layer ΔP, lb.f/sq. ft.	f_1	f_k
Cubic									
	$\epsilon = 0.480$ (packed)			$L = 6.15$ in.					
1	2.57	0.00744	2,960	0.459	23.73	1.312			
5	6.43	0.04673	7,460	2.410	19.83	1.096			
8	11.69	0.1518	13,310	7.014	17.77	0.983			
	$\epsilon = 0.729$ (distended)			$L = 7.59$ in.			$L = 1.52$ in.		
4	12.53	0.1745	27,160	1.300	4.454	0.863	0.2158	3.694	0.7157
5	7.13	0.0567	15,500	0.464	4.897	0.949	0.0816	4.295	0.8323
8	2.69	0.0814	5,900	0.0737	5.413	1.049	0.0134	4.915	0.9524
	$\epsilon = 0.882$ (distended)			$L = 9.90$ in.			$L = 1.98$ in.		
2	2.49	0.00687	12,370	0.0190	2.911	0.999	0.0030	2.299	0.7887
6	13.44	0.196	64,920	0.400	2.148	0.737	0.0716	1.923	0.6596
7	9.42	0.0979	46,280	0.217	2.334	0.801	0.0399	2.146	0.7360
Body-centered cubic									
	$\epsilon = 0.354$ (packed)			$L = 3.76$ in.					
1	9.90	0.1083	8,940	10.385	48.54	1.077			
5	5.43	0.0326	4,900	3.611	56.07	1.244			
8	2.81	0.00878	2,550	1.099	63.38	1.406			
	$\epsilon = 0.615$ (distended)			$L = 4.04$ in.			$L = 1.635$ in.		
1	2.56	0.00721	3,860	0.0806	8.842	1.028	0.0371	10.05	1.169
5	6.82	0.0518	10,440	0.4710	7.202	0.837	0.1966	7.410	0.862
8	11.44	0.1404	16,750	1.222	6.883	0.800	0.4277	5.941	0.692
	$\epsilon = 0.728$ (distended)			$L = 4.31$ in.			$L = 1.76$ in.		
2	10.09	0.1142	22,060	0.5209	4.787	0.923			
4	6.94	0.0540	15,150	0.2518	4.893	0.944			
8	2.63	0.0764	5,620	0.0434	5.960	1.149			
1*	12.27	0.1650	26,310				0.2587	4.029	0.777
2*	4.29	0.0205	9,335				0.0355	4.460	0.860
3*	2.59	0.00747	5,650				0.0141	4.838	0.933
Face-centered cubic									
	$\epsilon = 0.743$ (distended)			$L = 6.03$ in.			$L = 2.42$ in.		
1	11.26	0.1378	25,100	0.7521	4.334	0.888	0.2636	3.782	0.775
4	6.90	0.0524	15,600	0.3054	4.627	0.949	0.1060	4.000	0.820
8	2.77	0.00838	6,200	0.0583	5.524	1.132	0.0230	5.427	1.112

* Pressure drop data taken across test layer only.

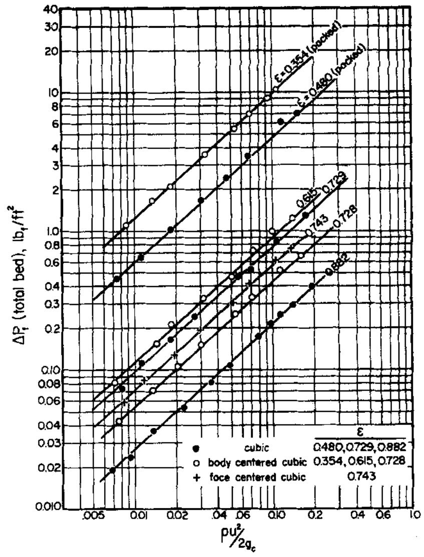

Fig. 2. Relationships between overall pressure drop and kinetic energy for the flow of air through packed and distended beds of spheres.

and 26 ft. high, which was connected to the suction end of a centrifugal blower. Two parallel perforated steel plates were placed 6 in. apart across the inlet of the wind tunnel to insure uniform air velocities throughout. A sliding window near the blower permitted the flow of air to be varied from 2.5 to 13.4 ft./sec. The test section containing the fixed bed was 22 in. high and was placed 10 ft. below the inlet of the wind tunnel. The air velocity was measured 26 in. above the test section with a sliding pitot tube which was connected to a micromanometer capable of detecting pressure differences of 0.0001 in. of manometer fluid (isobutyl alcohol).

The beds contained plastic phenolic spheres, 1.23 in. in diameter, which were arranged in a specific geometric configuration and held fixed in space with short lengths of fine, rigid wire, 0.018 in. in diameter. The wires were permanently fixed with epoxy resin into holes drilled into the spheres at the proper angles. The test beds were first shaped to the cylindrical geometry of the wind tunnel by removing excess portions of the external spheres with a mechanical saw to eliminate channeling near the wall. Because of the excessive weight of the solid plastic spheres the beds were suspended from the bottom with 35-lb. test nylon lines.

The packed and distended beds contained five layers of spheres and had the following geometric configurations and void fractions:

Orientation	ϵ
Cubic	0.480, 0.729, 0.882
Body-centered cubic	0.354, 0.615, 0.728
Face-centered cubic	0.743

A photograph of the distended bed, having a body-centered cubic arrangement and a void fraction $\epsilon = 0.728$, is presented in Figure 1.

The void fractions were determined by placing each bed into a cylindrical container into which water was then added up to a point in the bottom layer of spheres in the bed. A measured quantity of water was then added until the level reached a corresponding point in the top

Fig. 1. Distended fixed bed of spheres having a body-centered cubic arrangement ($d_s = 1.23$ in. and $\epsilon = 0.728$).

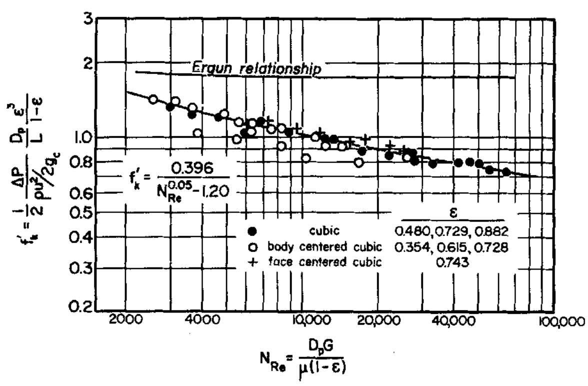

Fig. 3. Relationship between friction factor f'_k and modified Reynolds number resulting from overall pressure drop data of packed and distended beds of spheres.

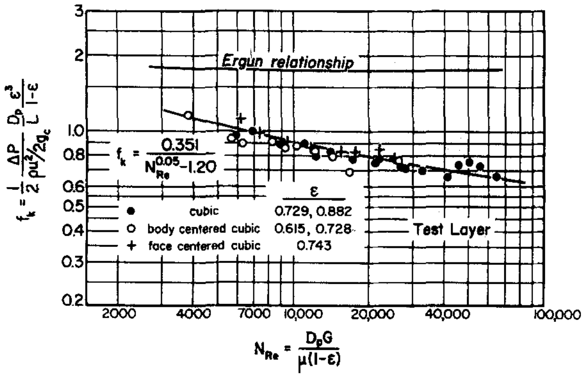

Fig. 4. Relationship between friction factor f_k and modified Reynolds number for distended beds of spheres (based on pressure drop across test layer of bed).

layer and the distance between the upper and lower points was determined. The ratio of the volume of water added to the product of the distance between the two points and the cross-sectional area of the bed produced ϵ.

The pressure drop through the bed was measured with static pressure taps mounted perpendicularly into the walls of the wind tunnel, 5 in. upstream and 7.5 ft. downstream from the test section. The lower tap contained a sliding probe which enabled the determination of the static pressure profile downstream from the bed. The pressure taps were connected to the micromanometer, from which the overall pressure drop could be determined.

Pressure drop measurements were also made across the middle layer of spheres in each distended bed investigated. A set of seven equally spaced static pressure taps provided with sliding probes was placed the same distance above and below the test layer. The spacing of the taps varied with the geometrical configuration and void fraction of the bed. The taps were connected to the micromanometer, from which the static pressure traverses above and below the test layer were determined. Similar measurements were not possible for the two packed beds investigated.

INTERPRETATION OF EXPERIMENTAL RESULTS

The drag force of a fluid exerted on the spheres of a bed $F = A_c\Delta P$, can be related to the area of the spheres and the kinetic energy of the fluid flowing past their surfaces as follows:

$$F = f A_s \frac{\rho u^2}{2 g_c} \qquad (1)$$

f includes contributions due to the pressure distribution over the surface and shear effects caused by velocity gradients. Since $A_s = a V = a A_c L$

$$\Delta P = f a L \frac{\rho u^2}{2 g_c} \qquad (2)$$

where a can be related to the diameter of the spheres D_p and void fraction of the bed as follows:

$$a = \frac{6}{D_p}(1-\epsilon) \qquad (3)$$

Combining Equations (2) and (3) one gets

$$\frac{\Delta P}{\rho u^2/2 g_c}\frac{D_p}{L}\frac{1}{1-\epsilon} = f_1 \qquad (4)$$

where $f_1 = 6f$ is a function of the modified Reynolds number. The bed height depends both on the geometric orientation and void fraction of the bed.

Overall Pressure Drop Measurements

Fifty-one experimental runs were conducted on the packed and distended beds of spheres having body-centered cubic and cubic arrangements to obtain information on the overall pressure drop in the beds. The experimental data for typical runs are presented in Table I. The experimental data for all the runs of this investigation may be found elsewhere (9). The resulting pressure drop measurements are plotted in Figure 2 vs. $\rho u^2/2 g_c$, the superficial kinetic energy of the flowing air. Separate parallel lines having slopes equal to 0.90 resulted for each void fraction. For beds having the same orientation the pressure drop was found to decrease with void fraction for a given kinetic energy. The overall pressure drop through the bed includes the entrance and exit effects of the air flowing through the bed.

Values of the friction factor f_1' were calculated from Equation (4) for each experimental run and were plotted against the corresponding modified Reynolds number $D_pG/\mu(1-\epsilon)$. Separate parallel curves resulted for each void fraction with the friction factor decreasing, while void fraction increased for a given modified Reynolds number.

Ergun (3) has shown that for randomly packed beds of spheres the following modified form of Equation (4) produces a single relationship between the friction factor f_k and the modified Reynolds number, which is independent of the void fraction of the bed:

$$\frac{\Delta P}{\rho u^2/g_c}\frac{D_p}{L}\frac{\epsilon^3}{1-\epsilon} = f_k \qquad (5)$$

Ergun determined the following relationship between the friction factor f_k and modified Reynolds number:

$$f_k = 1.75 + \frac{150}{N_{Re}} \qquad (6)$$

When values of the friction factor calculated with Equation (5) were plotted vs. the modified Reynolds number for each experimental run, a single curve resulted for both packed and distended beds, as shown in Figure 3. Some deviations are noted for the distended bed having a void fraction $\epsilon = 0.615$. Because of the consistency of the other values these deviations appear to be the result of some source of significant experimental error which was not present during the runs on the other beds. Despite this anomaly it is felt that generally the pressure drop measurements are reproducible to within the accuracy of the micromanometer. The relationship produced may be expressed analytically as follows:

$$f'_k = \frac{0.396}{N_{Re}^{0.05} - 1.20} \qquad (7)$$

The Ergun relationship, Equation (6), is also presented in Figure 3. From this figure it may be seen that the Ergun relationship produces higher

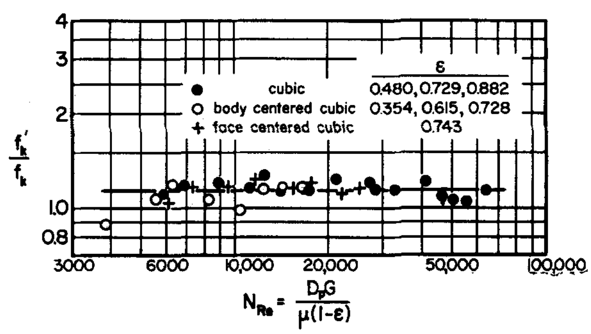

Fig. 5. Relationship between f'_k/f_k and modified Reynolds number for packed and distended beds having five layers of spheres.

values of the friction factor than those obtained from the experimental data of this study. This discrepancy can be attributed to the fact that in the present study spheres with very smooth surfaces were employed, and therefore the friction factors obtained were smaller than those resulting from Ergun's study in which spheres having varying degrees of roughness were utilized.

The fact that the single relationship of Equation (7) resulted for packed and distended beds having body-centered cubic and cubic arrangements of varying void fractions suggests that a single relationship between the friction factor f'_k and the modified Reynolds number exists for both types of beds which is independent of the geometric orientation and void fraction of the bed. In order to confirm this indication eight additional experimental runs were conducted with a distended bed whose spheres were arranged in a face-centered cubic orientation having a void fraction $\epsilon = 0.743$. The resulting experimental and calculated values are plotted in Figures 2 and 3. Data of typical runs are also presented in Table 1. Figure 3 shows that the friction factors calculated from Equation (5) for this face-centered cubic bed are identical to the corresponding values for the beds having the other two orientations. Therefore Equation (7) represents the relationship between f'_k and N_{Re} for packed and distended beds whose spheres are arranged in any type of geometric orientation.

Pressure Drop Measurements Across the Test Layer

For each distended bed the pressure drop across the middle layer of spheres was also measured to eliminate entrance and exit effects in the correlation of friction factor with modified Reynolds number. The resulting pressure drop measurements were subjected to an analysis similar to that used previously. The data of typical experimental runs are also presented in Table 1. Straight lines of slope 0.95 were obtained when the pressure drop across the test layer was plotted against the superficial kinetic energy of the flowing air, as compared with the lines of slope 0.90 which resulted from the overall pressure drop measurements. Values of the friction factor f_1 for the test layers were calculated for each experimental run from Equation (4) and were plotted against the corresponding Reynolds number. As before, separate parallel curves resulted for each void fraction investigated. Values of the friction factor f_k calculated with Equation (5) for the test layers were also plotted against the modified Reynolds number as shown in Figure 4. Again a single relationship resulted between f_k and N_{Re} for the body-centered cubic, face-centered cubic, and cubic types of distended beds. The resulting analytical relationship is

$$f_k = \frac{0.351}{N_{Re}^{0.05} - 1.20} \qquad (8)$$

When one divides Equation (7) by Equation (8), for any modified Reynolds number the friction factor resulting from the overall pressure drop across the five-layer beds is 1.13 times greater than the friction factor resulting from the pressure drop across the test layer. This point is illustrated in Figure 5, where values of the ratio f'_k/f_k obtained directly from experimental data are plotted against the corresponding modified Reynolds number. Larger friction factors are obtained for the overall pressure drops because of the entrance and exit effects which are inherent in these measurements and which appear to be independent of modified Reynolds number.

Conclusions of Results

It has been shown that a single relationship exists between f_k and N_{Re} for packed and distended beds of spheres having any type of geometric arrangement. If friction factors f_k are desired, in which entrance and exit effects of the air flowing through the bed are eliminated, Equation (8) represents the correct dependence between these friction factors and modified Reynolds number. The corresponding pressure drop of the bed can then be calculated from Equation (5). However if pressure drops are required in which entrance and exit effects are included, Equation (7) represents the correct dependence between the corresponding friction factors f'_k and the modified Reynolds number only for packed and distended beds having five layers of spheres.

As the number of layers of spheres in the bed increases, it is to be expected that the ratio f'_k/f_k should decrease from the value of 1.13 obtained for five layers and approach the limiting value of 1 as the number of layers becomes very large, because entrance and exit effects are negligible compared with the pressure drop of the bed when the number of layers is large. Thus for beds containing many layers of spheres, as commonly encountered in practice, Equation (8) should accurately represent the relationship between the friction factor f_k and modified Reynolds number. For beds of spheres having less than five layers the ratio f'_k/f_k should be greater than 1.13, reaching a maximum value for a single layer of spheres. Additional experimental studies are required to obtain pressure drops through beds of varying height, so the exact relationship between the ratio f'_k/f_k and the number of layers of spheres can be established.

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NOTATION

a = surface area of bed, sq.ft./cu.ft.
A_c = cross-sectional area of bed, sq. ft.
A_s = surface area of spheres in bed, sq.ft.
d_s = sphere diameter, in.
D_p = sphere diameter, ft.
f = friction factor, Equation (1)
f'_1 = friction factor based on entire bed height, Equation (4)
f_1 = friction factor based on test layer, Equation (4)
f'_k = friction factor based on entire bed height, Equation (5)
f_k = friction factor based on test layer, Equation (5)
F = drag force, lb.$_f$
G = superficial mass velocity, lb.$_m$/sec. sq.ft.
g_c = conversion factor, 32.17 lb.$_m$ ft./lb.$_f$ sec^2
L = bed height, ft.
N_{Re} = modified Reynolds number, $D_pG/\mu(1-\epsilon)$
ΔP = pressure drop, lb$_f$/sq.ft.
u = superficial gas velocity, ft./sec.
V = volume of bed. cu.ft.
ϵ = void fraction
μ = absolute viscosity, lb.$_m$/sec.ft.
ρ = density, lb.$_m$/cu.ft.

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